Worcester County Mathematics League

Varsity Meet 2 - November 29, 2023

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 2 - November 29, 2023
Answer Key

Round 1 - Fractions, Decimals, and Percents

1. $-\dfrac{49}{40}$
2. 24
3. $\dfrac{1}{2}, 3$ need both, either order (or $0.5, 3$)

Round 2 - Algebra I

1. $\dfrac{5}{12}$
2. $\dfrac{1}{1-a}$
3. $\dfrac{64}{81}$

Round 3 - Parallel Lines and Polygons

1. 75
2. 22.5
3. $\dfrac{3+\sqrt{3}}{3}$ or $1+\frac{\sqrt{3}}{3}$

Round 4 - Sequences and Series

1. 13
2. $(2, 9, 16)$, exact order
3. $\dfrac{428}{105}$

Round 5 - Matrices and Systems of Equations

1. 4
2. 6713 or $6713
3. $\dfrac{5}{2}$

Team Round

1. 75
2. -55
3. 21
4. 750500 or 750,500
5. $\begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix}$
6. 11268 or $11268
7. 320
8. 1440
9. $\frac{1}{2}$ or 0.5

Worcester County Mathematics League

Varsity Meet 2 - November 29, 2023

Round 1 - Fractions, Decimals, and Percents

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Simplify the following expression. Write your answer as a fraction (possibly improper) such as $\frac{9}{7}$ or $-\frac{2}{5}$.

$$\left(-\frac{1}{2}\right)^3 - \left[\left(\frac{2}{3}+\frac{1}{4}\right) \div \frac{5}{6}\right]$$

2. In a certain dynasty, one third of the queens were named Anne, 25% were named Beatrice, 0.125 were named Cynthia, one twelfth were named Doreen, and five had other names. How many queens were there in total?

3. Solve for x.

$$\frac{\frac{1}{x}+\frac{1}{x+1}}{\frac{1}{x}+\frac{1}{x+3}} = \frac{7}{6}$$

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. $x =$ ______________________

Shepherd Hill, Bromfield, Worcester Academy

Worcester County Mathematics League

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Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Solve the following equation for x. Express your answer in the form $\frac{n}{m}$ (possibly an improper fraction).

$$1 - 2\left(2 - 3\left(3 - 4x\right)\right) = 5$$

2. Simplify the following rational expression:

$$\cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{1-a}}}}$$

3. Find the product xy given the following two equations:

$$\begin{aligned} x + y &= \frac{20}{9} \\ \sqrt{x} - \sqrt{y} &= \frac{2}{3} \end{aligned}$$

<u>**ANSWERS**</u>

(1 pt) 1. $x =$ ____________________

(2 pts) 2. ____________________

(3 pts) 3. ____________________

Notre Dame, South, Worcester Academy

Worcester County Mathematics League

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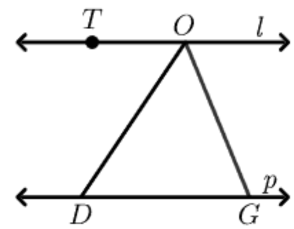
Round 3 - Parallel Lines and Polygons

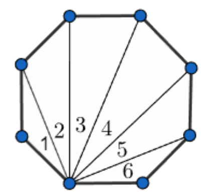
All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

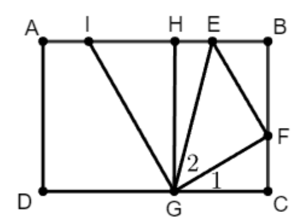
1. Given $l||p$, $\text{m}\angle DOT = 43°$ and $\text{m}\angle DOG = 62°$ in the figure at the right, find $\text{m}\angle DGO$.



2. The polygon shown in the figure to the right is a regular octagon. Note that five diagonals are drawn from a vertex, dividing the octagon into six triangles. Find the largest degree measure among the six angles $\angle 1$, $\angle 2$, ..., $\angle 6$. Express your answer as a decimal, in degrees.



3. Given rectangle $ABCD$ in the figure at the right, with $m\angle 1 = 30°$, $m\angle 2 = 45°$, $\overline{EF} \perp \overline{GF}$, $\overline{GH} \perp \overline{AB}$, $\overline{IG}||\overline{EF}$, and $FC = 1$. Find IH. Express your answer in simplest radical form.



ANSWERS

(1 pt) 1. ______________________ °

(2 pts) 2. ______________________ °

(3 pts) 3. ______________________

St Peter Marian, Quaboag, Auburn

Worcester County Mathematics League
Varsity Meet 2 - November 29, 2023
Round 4 - Sequences and Series

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. In the arithmetic sequence $-16, -9, -2, \ldots$, where -16, -9, and -2 are terms 1, 2, and 3 of the sequence, what is the term number of 68?

2. If l, m, and n are three consecutive terms of an arithmetic sequence where $l + m + n = 27$ and $l \cdot m \cdot n = 288$, find the ordered triple (l, m, n), where $l < m < n$.

3. A sequence is harmonic if each term in the sequence is the harmonic mean of its two neighbors. Note that s is the harmonic mean of r and t if $\frac{1}{s} = \frac{\frac{1}{r} + \frac{1}{t}}{2}$. Given the harmonic sequence $1, a, b, c, 2, \ldots$, find $a + b + c$. Write your answer as an improper fraction in the form $\dfrac{m}{n}$.

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. $(l, m, n) = ($____________$)$

(3 pts) 3. $a + b + c =$ ____________

Oxford, South, Worcester Academy

Round 5 - Matrices and Systems of Equations

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Determine the value of k such that the two lines $2x + ky = 8$ and $3x - (k+1)y = 1$ intersect at point $(x, y) = (2, 1)$.

2. The matrix shown at right represents attendance at three separate performances of a South HS play, where the first column represents the number of adults in attendence, the second column represents the number of children in attendance, and the three rows represent the attendance at the Thursday, Friday and Saturday performances. If the price of an adult's ticket is \$5 and the cost of a child's ticket is \$3, what was the total amount of money of ticket sales for the three performances?

$$\begin{bmatrix} 123 & 201 \\ 342 & 526 \\ 217 & 374 \end{bmatrix}$$

3. Solve for x:

$$\begin{vmatrix} 2x & -2 & 5 \\ 4 & 3 & -2x \\ 3x & 1 & 0 \end{vmatrix} = 35x - 80$$

ANSWERS

(1 pt) 1. $k =$ ____________________

(2 pts) 2. \$ ____________________

(3 pts) 3. $x =$ ____________________

Doherty, South, Shrewsbury

Worcester County Mathematics League

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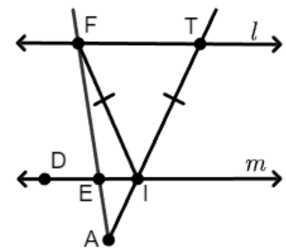
Team Round

All answers must be in simplest exact form in the answer section.

NO CALCULATORS ALLOWED

1. Each day, Julie eats 20% of the cookies that were in her cookie jar at the start of the day. At the end of the second day, 48 cookies remain in the jar. How many cookies were in the jar at the start of the first day?

2. Four consecutive odd integers have a sum of -208. Find the smallest of these integers.



3. Given the figure at the right, with $l||m$, $\overline{IF} \cong \overline{IT}$, $m\angle DEF = 87°$, and $m\angle FIT = 36°$, find $m\angle FAT$.

4. Find the sum of all even integers from -1000 to 2000, inclusive.

5. Multiply:

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

.

6. Kevin loans Bill $10,000. He charges Bill 1% interest at the end of each month. If Bill makes no payments to Kevin, how much money does he owe Kevin at the end of one year? Round your answer to the nearest dollar.

7. A temperature's measurements in degrees Fahrenheit (F) and degrees Celsius (C) are related by the formula $F = \frac{9}{5}C + 32$. At what temperature is the degrees Fahrenheit exactly twice the number of degrees Celsius? Express your answer in degrees Fahrenheit.

8. Five regular polygons fit together like a puzzle; they share a vertex V and completely fill the plane around V without overlap. Find the largest possible sum of the measures of all of the angles of the five polygons, in degrees.

9. Find the value of the following infinite sum:

$$\sum_{n=2}^{\infty} \frac{1}{n^2 + n}$$

Ashland, Tantasqua, Quaboag, Tantasqua, North, AMSA, Anon., Worc. Acad., Quaboag

Worcester County Mathematics League
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Team Round Answer Sheet

ANSWERS

1. ________________________

2. ________________________

3. ________________________°

4. ________________________

5. $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

6. $ ______________________

7. ________________________°F

8. ________________________°

9. ________________________

Worcester County Mathematics League

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6. 11268 or \$11268
7. 320
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Round 1 - Fractions, Decimals, and Percents

1. Simplify the following expression. Write your answer as a fraction (possibly improper) such as $\frac{9}{7}$ or $-\frac{2}{5}$.

$$\left(-\frac{1}{2}\right)^3 - \left[\left(\frac{2}{3}+\frac{1}{4}\right) \div \frac{5}{6}\right]$$

Solution:
$\left(-\frac{1}{2}\right)^3 - \left[\left(\frac{2}{3}+\frac{1}{4}\right) \div \frac{5}{6}\right] = \frac{(-1)^3}{2^3} - \left[\left(\frac{8}{12}+\frac{3}{12}\right) \cdot \frac{6}{5}\right] = -\frac{1}{8} - \left[\frac{11}{12} \cdot \frac{6}{5}\right] = -\frac{1}{8} - \frac{11}{10} = -\frac{5}{40} - \frac{44}{40} = \boxed{-\frac{49}{40}}$

2. In a certain dynasty, one third of the queens were named Anne, 25% were named Beatrice, 12.5% were named Cynthia, one twelfth were named Doreen, and five had other names. How many queens were there in total?

Solution: Represent each of the given values as fractions: $\dfrac{1}{3}$ of the queens were Anne, $\dfrac{1}{4}$ were Beatrice, $\dfrac{1}{8}$ were Cynthia, and $\dfrac{1}{12}$ were Doreen. The least common denominator is 24, and the sum of those fractions is $\dfrac{1}{3}+\dfrac{1}{4}+\dfrac{1}{8}+\dfrac{1}{12} = \dfrac{8}{24}+\dfrac{6}{24}+\dfrac{3}{24}+\dfrac{2}{24} = \dfrac{19}{24}$. This means that the remaining five queens with other names made up the remaining $\dfrac{24}{24} - \dfrac{19}{24} = \dfrac{5}{24}$ portion of the queens, so there were $\boxed{24}$ total queens.

3. Solve for x:

$$\frac{\frac{1}{x}+\frac{1}{x+1}}{\frac{1}{x}+\frac{1}{x+3}} = \frac{7}{6}$$

Solution: First, multiply the numerator and the denominator of the left side by x, and represent each of the numerator and denominator as a single fraction, yielding $\dfrac{\frac{1}{x}+\frac{1}{x+1}}{\frac{1}{x}+\frac{1}{x+3}} = \frac{1+\frac{x}{x+1}}{1+\frac{x}{x+3}} = \dfrac{\frac{x+1}{x+1}+\frac{x}{x+1}}{\frac{x+3}{x+3}+\frac{x}{x+3}} = \dfrac{\frac{2x+1}{x+1}}{\frac{2x+3}{x+3}} \left(=\frac{7}{6}\right)$. Replace division a fraction with multiplication by its reciprocal: $\dfrac{\frac{2x+1}{x+1}}{\frac{2x+3}{x+3}} = \dfrac{(2x+1)(x+3)}{(2x+3)(x+1)} \left(=\frac{7}{6}\right)$. Cross multiply: $6(2x+1)(x+3) = 7(2x+3)(x+1)$. Expand the expressions in parentheses: $12x^2+42x+18 = 14x^2+35x+21$. Combine like terms on the right $0 = 2x^2-7x+3$, and factor: $2x^2-7x+3 = (2x-1)(x-3) = 0$. The two roots are: $x = \boxed{\dfrac{1}{2}, 3}$.

Round 2 - Algebra I

1. Solve the following equation for x. Express your answer as a (possibly improper) fraction in the form $\frac{n}{m}$.

$$1 - 2\left(2 - 3\left(3 - 4x\right)\right) = 5$$

Solution: First simplify the equation starting with the innermost bracket:

$$-3\left(3 - 4x\right) = -9 + 12x$$

using the distributive property, where -3 is distributed (*including the sign*) over the sum $3 - 4x$. Now add 2 to this expression and use the distributive property to eliminate the outermost bracket:

$$-2\left(2 - 3\left(3 - 4x\right)\right) = -2\left(2 - 9 + 12x\right) = -2\left(-7 + 12x\right) = 14 - 24x$$

Next, add 1 to this expression and set it equal to 5:

$$1 - 2\left(2 - 3\left(3 - 4x\right)\right) = 1 + 14 - 24x = 15 - 24x = 5$$

Finally, add $24x$ and subtract 5 from both sides of the equation, so $10 = 24x$ and $x = \frac{10}{24} = \boxed{\dfrac{5}{12}}$ when simplified.

2. Simplify the following rational expression:

$$\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{1-a}}}}$$

Solution: Begin the simplification process from the bottom up:

$$1-\frac{1}{1-a}=\frac{1-a}{1-a}-\frac{1}{1-a}=\frac{1-a-1}{1-a}=\frac{-a}{1-a}$$

Move one level up on the continued fraction:

$$1-\frac{1}{1-\frac{1}{1-a}}=1-\frac{1}{\frac{-a}{1-a}}=1-\frac{1-a}{-a}=\frac{a}{a}+\frac{1-a}{a}=\frac{a+1-a}{a}=\frac{1}{a}$$

Now move another level up:

$$1-\frac{1}{1-\frac{1}{1-\frac{1}{1-a}}}=1-\frac{1}{\frac{1}{a}}=1-a$$

The original expression is just the reciprocal of this expression, so the final simplification is $\boxed{\dfrac{1}{1-a}}$.

3. Find the product xy given the following two equations:

$$x+y=\frac{20}{9}$$
$$\sqrt{x}-\sqrt{y}=\frac{2}{3}$$

Solution: The simplest solution to this system of equations begins by squaring both sides of the second equation:

$$\left(\sqrt{x}-\sqrt{y}\right)^2=\left(\frac{2}{3}\right)^2$$

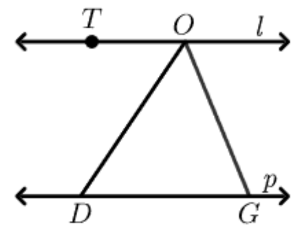
Now expand the left hand side using the identity $(a-b)^2=a^2-2ab+b^2$ and note that $\left(\frac{2}{3}\right)^2=\frac{2^2}{3^2}=\frac{4}{9}$:

$$\left(\sqrt{x}+\sqrt{y}\right)^2=\left(\sqrt{x}\right)^2-2\sqrt{x}\sqrt{y}+\left(\sqrt{y}\right)^2=x-2\sqrt{xy}+y=x+y-2\sqrt{xy}=\frac{4}{9}$$

Next, substitute $\frac{20}{9}$ for $x+y$ so that $\frac{20}{9}-2\sqrt{xy}=\frac{4}{9}$. Add $2\sqrt{xy}-\frac{4}{9}$ to both sides of the equation and $2\sqrt{xy}=\frac{20}{9}-\frac{4}{9}=\frac{16}{9}$. Then $\sqrt{xy}=\frac{8}{9}$ and $xy=\left(\frac{8}{9}\right)^2=\boxed{\dfrac{64}{81}}$.

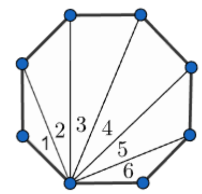
Round 3 - Parallel Lines and Polygons

1. Given $l||p$, $m\angle DOT = 43°$ and $m\angle DOG = 62°$ in the figure at the right, find $m\angle DGO$.

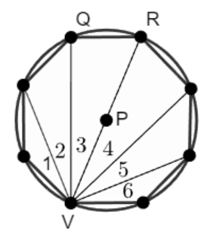


Solution: Since $m\angle DOT = 43°$ and $m\angle DOG = 62°$, the third acute angle at vertex O must have measure $180° - m\angle DOT - m\angle DOG = 180° - 43° - 62° = 75°$. By alternate interior angles, this angle is congruent to $\angle DGO$, and thus $m\angle DGO = \boxed{75}°$.

2. The polygon shown in the figure to the right is a regular octagon. Note that five diagonals are drawn from a vertex, dividing the octagon into six triangles. Find the largest degree measure of the six angles $\angle 1, \angle 2, \ldots, \angle 6$. Express your answer as a decimal, in degrees.



Solution: First, note that a regular octagon, in fact any regular polygon, can be inscribed in a circle. The circle for this octagon, with center P, is shown in the figure at the right, with three vertices labeled V, Q and R. Recall that congruent chords in a circle have congruent arcs. The eight sides of the octogon are congruent and each side is a chord of circle P.



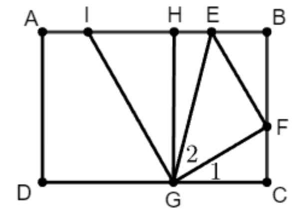
Solution: (continued) Thus, the eight arcs defined by consecutive vertices of the octogon, for example $\overset{\frown}{QR}$, are all congruent and have equal measure. Since the measure of a circle is 360°, the measure of any individual arc is $360/8 = 45°$. Now each of the six angles shown with vertex V is an inscribed angle that intercepts a 45° arc. Recall that the measure of an inscribed angle is half the measure of its intercepted arc. Therefore each angle has measure $45/2 = \boxed{22.5}°$, which is therefore the largest angle measure.

Alternative Solution: Identify the six triangles by the numbered angles $1, 2, 3, 4, 5, 6$. Note that $\triangle 1$ and $\triangle 6$ are congruent isosceles triangles, each with an interior angle of the octagon of measure 135°. The other two angles have equal measures x, and the sum of the measures of angles in a triangle is 180°, so $2x + 135 = 180$ and $x = (180 - 135) \div 2 = \frac{45}{2}$. Thus $m\angle 1 = m\angle 6 = 22.5°$.

Then $\triangle 3$ and $\triangle 4$ are congruent by symmetry and $m\angle 3 = m\angle 4$. Also, opposite sides of the octogon are parallel, so $\overline{VQ} \perp \overline{QR}$ and $\triangle 3$ and $\triangle 4$ are right triangles. In $\triangle 3$, $m\angle VRQ = 135 \div 2 = 67.5°$ because $\overline{VR}$ bisects an internal angle of the octogon. Note that $\angle VRQ$ is complementary to $\angle 3$, so $m\angle 3 = 90 - m\angle VRQ = 90 - 67.5 = 22.5° = m\angle 4$.

Finally, $\triangle 2$ and $\triangle 5$ are congruent, and angles $1, 2, 3, 4, 5, 6$ form an interior angle of measure 135°, so $2m\angle 2 = 2m\angle 5 = 135 - 4 \cdot 22.5 = 135 - 90 = 45°$, and each of the six angles has measure 22.5°.

3. Given rectangle $ABCD$ in the figure at the right, with $m\angle 1 = 30°$, $m\angle 2 = 45°$, $\overline{EF} \perp \overline{GF}$, $\overline{GH} \perp \overline{AB}$, $\overline{IG}||\overline{EF}$, and $FC = 1$. Find IH. Express your answer in simplest radical form.



Solution: Note that $\triangle FCG$ is a 30°-60°-90° triangle because m$\angle CGF = 30°$ and m$\angle FCG = 90°$. Recall that the lengths of the sides opposite the 30°, 60°, and 90° angles of such a triangle are in proportion $1 : \sqrt{3} : 2$, in that order. Since $FC = 1$ and is opposite the 30° angle, the hypotenuse $\overline{FG}$ must be length 2. Similarly, as $\triangle EFG$ is a right triangle with a 45° angle, it must be right isosceles, so $EF = FG = 2$.

As m$\angle FGC = 30°$ and $\angle C$ is right, it follows that m$\angle GFC = 90° - 30° = 60°$, and that m$\angle EFB = 180° - m\angle EFG - m\angle GFC = 180° - 90° - 60° = 30°$. Thus $\triangle EBF$ is also a 30°-60°-90° triangle. It is congruent to $\triangle FCG$ because their hypotenuses are congruent. Therefore, from the stated proportion, $BF = \sqrt{3}$. And since $HBCG$ is a rectangle, $GH = BC = FC + FB = 1 + \sqrt{3}$.

Now $\overline{IG}||\overline{EF}$ and $\overline{GH}||\overline{FB}$, so $\angle EFB$ and $\angle IGH$ are corresponding angles and m$\angle IGH =$ m$\angle EFB = 30°$. Thus $\triangle IHG$ is also 30°-60°-90°, and $IH = \frac{1}{\sqrt{3}} \cdot GH = \frac{\sqrt{3}}{3} \cdot (1 + \sqrt{3}) = \boxed{\dfrac{3+\sqrt{3}}{3}}$.

Round 4 - Sequences and Series

1. In the arithmetic sequence $-16, -9, -2, \ldots$, where -16, -9, and -2 are terms 1, 2, and 3 of the sequence, what is the term number of 68?

Solution: The common difference of this sequence is $-9-(-16) = -9+16 = 7$, that is, successive terms increase by 7. The difference between 68 and the first term is $68 - (-16) = 84$.

Note that the common difference must be added to the first term $n-1$ times to reach the n^{th} term. In this sequence the common difference (7) is added to the first term (-16) twelve times to reach 68 because $\frac{84}{7} = 12$. If the term number of 68 is n, then $12 = n - 1$, and $n = \boxed{13}$.

2. If l, m, and n are three consecutive terms of an arithmetic sequence where $l + m + n = 27$ and $l \cdot m \cdot n = 288$, find the ordered triple (l, m, n), where $l < m < n$.

Solution: If the the three terms are ordered l, m, and n and the common difference of the sequence is d, then $(l, m, n) = (m-d, m, m+d)$. Then $l+m+n = m-d+m+m+d = m+m+m+d-d = 3m = 27$, or $m = 9$. Next, find d, which must be positive since the sequence is increasing. From the second condition, $l \cdot m \cdot n = (m-d) \cdot m \cdot (m+d) = 9 \cdot (9-d) \cdot (9+d) = 9 \cdot (9^2 - d^2) = 288$, where the known value $m = 9$ is substituted in the third step and the difference of squares identity is applied in the last step. Next, $81 - d^2 = 288 \div 9 = 32$, so $d^2 = 81 - 32 = 49$, and $d = 7$. Finally, $l = m - d = 9 - 7 = 2$ and $n = m + d = 9 + 7 = 16$, so $(l, m, n) = \boxed{(2, 9, 16)}$.

3. A sequence is harmonic if each term in the sequence is the harmonic mean of its two neighbors. Note that s is the harmonic mean of r and t if $\frac{1}{s} = \dfrac{\frac{1}{r} + \frac{1}{t}}{2}$. Given the harmonic sequence $1, a, b, c, 2, \ldots$, find $a + b + c$. Write your answer as an improper fraction in the form $\dfrac{m}{n}$.

Solution: A simple solution results from the observation that if l, m, n form a harmonic sequence, then $\frac{1}{l}, \frac{1}{m}, \frac{1}{n}$ form an arithmetic sequence. That fact is justified as follows: $\frac{1}{m} = \dfrac{\frac{1}{l} + \frac{1}{n}}{2}$, so $\frac{2}{m} = \frac{1}{m} + \frac{1}{m} = \frac{1}{l} + \frac{1}{n}$ and $\frac{1}{m} - \frac{1}{l} = \frac{1}{n} - \frac{1}{m} = d$, where d is the common difference. Apply this fact to the given harmonic sequence. Then $1, \frac{1}{a}, \frac{1}{b}$ form an arithmetic sequence with common difference d. Also $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ form an arithmetic sequence with the same common difference $d = \frac{1}{a} - 1 = \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$. Likewise, $\frac{1}{b}, \frac{1}{c}, \frac{1}{2}$ form an arithmetic sequence with common difference d, and so $1, \frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{2}$ form an arithmetic sequence. There are four differences from 1 to $\frac{1}{2}$, so $d = \frac{\frac{1}{2}-1}{4} = -\frac{1}{8}$. The arithmetic sequence $1, \frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{2} = 1, 1 - \frac{1}{8}, 1 - \frac{2}{8}, 1 - \frac{3}{8}, 1 - \frac{1}{2} = 1, \frac{7}{8}, \frac{3}{4}, \frac{5}{8}, \frac{1}{2}$. Then $\frac{1}{a} = \frac{7}{8}$, $\frac{1}{b} = \frac{3}{4}$, $\frac{1}{c} = \frac{5}{8}$, so that $(a, b, c) = \left(\frac{8}{7}, \frac{4}{3}, \frac{8}{5}\right)$. Then $a + b + c = \frac{8}{7} + \frac{4}{3} + \frac{8}{5} = \frac{8 \cdot 15}{105}, \frac{4 \cdot 35}{105}, \frac{8 \cdot 21}{105} = \frac{120+140+168}{105} = \boxed{\dfrac{428}{105}}$

Round 5 - Matrices and Systems of Equations

1. Determine the value of k such that the two lines $2x + ky = 8$ and $3x - (k+1)y = 1$ intersect at point $(x, y) = (2, 1)$.

Solution: Plug in the values $x = 2$, $y = 1$ into the first equation: $2 \cdot 2 + k \cdot 1 = 8$, $4 + k = 8$ and $k = \boxed{4}$.

Similarly, plug in $x = 2$, $y = 1$ into the second equation so that $3 \cdot 2 - (k+1) \cdot 1 = 8$, $6 - (k+1) = 1$, $k + 1 = 6 - 1 = 5$, and again $k = 5 - 1 = 4$.

2. The matrix shown at right represents attendance at three separate performances of a South HS play, where the first column represents the number of adults in attendence, the second column represents the number of children in attendance, and the three rows represent the attendance at the Thursday, Friday and Saturday performances. If the price of an adult ticket is \$5 and the cost of a child's ticket is \$3, what was the total amount of money of ticket sales for the three performances?

$$\begin{bmatrix} 123 & 201 \\ 342 & 526 \\ 217 & 374 \end{bmatrix}$$

Solution: Let a be the cost of an adult's ticket, c be the cost of a child's ticket, T, F, and S be the total sales at Thursday's, Friday's performance, and Saturday's performances. Then the ticket sales for the three performances can be represented by the following system of equations:

$$\begin{cases} 123a + 201c = T \\ 342a + 526c = F \\ 217a + 374c = S \end{cases}$$

The question asks for the total sales of the three performances, or $T + F + S = x$. Therefore add these three equations to find x:

$$\begin{aligned} 123a + 342a + 217a + 201c + 526c + 374c &= x \\ (123 + 342 + 217)a + (201 + 526 + 374)c &= x \\ 682a + 1101c &= x \end{aligned}$$

Substitute $a = 5$ and $c = 3$: $x = 682 \cdot 5 + 1101 \cdot 3 = 3410 + 3303 = \$\boxed{6713}$.

3. Solve for x:

$$\begin{vmatrix} 2x & -2 & 5 \\ 4 & 3 & -2x \\ 3x & 1 & 0 \end{vmatrix} = 35x - 80$$

Solution: The determinant $\begin{vmatrix} 2x & -2 & 5 \\ 4 & 3 & -2x \\ 3x & 1 & 0 \end{vmatrix}$ evaluates to $2x \cdot 3 \cdot 0 + (-2) \cdot (-2x) \cdot 3x + 5 \cdot 4 \cdot 1 - 5 \cdot 3 \cdot 3x - (-2) \cdot 4 \cdot 0 - 2x \cdot (-2x) \cdot 1 = 12x^2 + 20 - 45x - (-4x^2) = 16x^2 - 45x + 20$. Setting this determinant equal to $35x - 80$ gives $16x^2 - 45x + 20 = 35x - 80$ which results in the quadratic equation $16x^2 - 80x + 100 = 0$ once every term is moved to the left. Each term is divisible by 4, so the quadratic is equivalent to $4x^2 - 20x + 25 = 0$ which factors nicely as $(2x - 5)^2 = 0$. Thus $2x - 5 = 0$, $2x = 5$, and the only solution is $x = \boxed{\frac{5}{2}}$.

Team Round

1. Each day, Julie eats 20% of the cookies that were in her cookie jar at the start of the day. At the end of the second day, 48 cookies remain in the jar. How many cookies were in the jar at the start of the first day?

Solution: Let x be the number of cookies in the jar at the start of the first day. If Julie eats 20% of the cookies in a given day, then there will be 80%, or $\frac{4}{5}$ of the cookies remaining at the end of the day. Then $\frac{4}{5}$ of the cookies from the end of the first day remain at the end of the second day, which is equal to $\frac{4}{5}$ of $\frac{4}{5}$ of the cookies at the start of the first day. Therefore $\frac{4}{5}\left(\frac{4}{5}x\right) = 48$, or $x = \frac{5}{4}\left(\frac{5}{4}48\right) = \frac{5}{4}(5 \cdot 12) = 5 \cdot 5 \cdot 3 = \boxed{75}$.

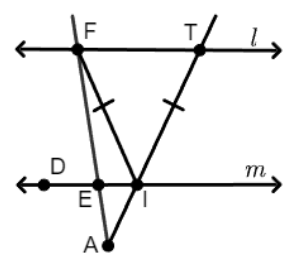
2. Four consecutive odd integers have a sum of -208. Find the smallest of these integers.

Solution: Note that consecutive odd integers increase by 2. Then if x is the smallest of the four integers:

$$x + x + 2 + x + 4 + x + 6 = -208$$

and therefore $4x + 12 = -208$, $4x = -208 - 12 = -220$, and $x = -220 \div 4 = \boxed{-55}$.

3. Given the figure at the right, with $l || m$, $\overline{IF} \cong \overline{IT}$, $m\angle DEF = 87°$, and $m\angle FIT = 36°$, find $m\angle FAT$.



Solution: First, since $\overline{IF} \cong \overline{IT}$, $\angle IFT \cong \angle ITF$ by the Isosceles Triangle Theorem. Since the sum of the angles in this triangle is 180°, $180° = m\angle FIT + 2m\angle IFT = 36 + 2m\angle IFT$. Then $180 - 36 = 144° = 2m\angle ITF$ and $m\angle ITF = 72°$.

Next, $\angle DEF \cong \angle AFT$ because they are alternate interior angles, so $m\angle AFT = m\angle DEF = 87°$. Finally, since the sum of the interior angles in $\triangle FAT$ is 180°, $m\angle FAT = 180° - m\angle AFT - m\angle ITF = 180 - 87 - 72 = \boxed{21}^{\circ}$

4. Find the sum of all even integers from -1000 to 2000, inclusive.

Solution: Let S represent the sum of all even integers from -1000 to 2000. Note that the sum of all even integers from -1000 to 1000 is zero, since the sum of -1000 to -2 is just the negative of the sum of 2 to 1000 in reverse order. So, S is the sum of all even integers from 1002 to 2000. This series has 500 terms, and is an arithmetic series. Apply the formula for an arithmetic series with $a_1 = 1002$, $a_n = 2000$, and $n = 500$:

$$S = \frac{n\,(a_1 + a_n)}{2} = \frac{500\,(1002 + 2000)}{2} = \frac{500 \cdot 3002}{2} = 500 \cdot 1501 = \boxed{750500}$$

5. Multiply:

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

.

Solution: Multiply the first two matrices.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{bmatrix} = \begin{bmatrix} a\cdot 0 + 0\cdot 0 + 0\cdot c & a\cdot 0 + 0\cdot b + 0\cdot 0 & a\cdot a + 0\cdot 0 + 0\cdot 0 \\ 0\cdot 0 + b\cdot 0 + 0\cdot c & 0\cdot 0 + b\cdot b + 0\cdot 0 & 0\cdot a + b\cdot 0 + 0\cdot 0 \\ 0\cdot 0 + 0\cdot 0 + c\cdot c & 0\cdot 0 + 0\cdot b + c\cdot 0 & 0\cdot a + 0\cdot 0 + c\cdot 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & a^2 \\ 0 & b^2 & 0 \\ c^2 & 0 & 0 \end{bmatrix}$$

Now multiply the resulting product matrix on its right side by $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ to yield the answer.

$$\begin{bmatrix} 0 & 0 & a^2 \\ 0 & b^2 & 0 \\ c^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0\cdot 0 + 0\cdot 0 + a^2\cdot 1 & 0\cdot 0 + 0\cdot 1 + a^2\cdot 0 & 0\cdot 1 + 0\cdot 0 + a^2\cdot 0 \\ 0\cdot 0 + b^2\cdot 0 + 0\cdot 1 & 0\cdot 0 + b^2\cdot 1 + 0\cdot 0 & 0\cdot 1 + b^2\cdot 0 + 0\cdot 0 \\ c^2\cdot 0 + 0\cdot 0 + 0\cdot 1 & c^2\cdot 0 + 0\cdot 1 + 0\cdot 0 & c^2\cdot 1 + 0\cdot 0 + 0\cdot 0 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix}}$$

Alternative solution: Note that $\begin{bmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = AT$.

Thus the desired product is $A(AT)T = (AA)(TT) = A^2$ since $T^2 = I$.

6. Kevin loans Bill \$10,000. He charges Bill 1% interest at the end of each month. If Bill makes no payments to Kevin, how much money does he owe Kevin at the end of one year? Round your answer to the nearest dollar.

Solution: If A is the total amount that Bill owes Kevin at the end of one year, then A can be found by multiplying \$10000 by 1.01 twelve times, or $A = 10000(1.01)^{12}$. The easiest way to approximate $(1.01)^{12}$ by hand is to use the Binomial Theorem:

$$A = (1.01)^{12} = (1 + .01)^{12} =_{12} C_0(1^{12}) +_{12} C_1(1^{11})(.01) +_{12} C_2 1^{10}(.01)^2 + \ldots +_{12} C_{12}(.01)^{12}$$

Note that the exponential $(.01)^n$ decreases very quickly, and much more quickly than the combination ${}_{12}C_n$ increases, which means that only a few terms of the binomial expansion need to be calculated to round to the nearest dollar. Also, the powers of 1 have no effect on the calculation and can be ignored. The first five terms of ${}_{12}C_n$ are calculated below:

$$\begin{aligned}
{}_{12}C_0 &= 1 \\
{}_{12}C_1 &= 12 \\
{}_{12}C_2 &= \frac{12 \cdot 11}{2} = 6 \cdot 11 = 66 \\
{}_{12}C_3 &= \frac{12 \cdot 11 \cdot 10}{3 \cdot 2} = 2 \cdot 11 \cdot 10 = 220 \\
{}_{12}C_4 &= \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2} = 11 \cdot 5 \cdot 9 = 495
\end{aligned}$$

A can therefore be approximated with the first 5 terms of the binomial:

$$\begin{aligned}
A =& 10000(1 +_{12} C_1(.01) +_{12} C_2(.01)^2 +_{12} C_3(.01)^3 +_{12} C_4(.01)^4 + \ldots) \\
=& 10000(1 + 0.12 + .0066 + .000220 + .00000495 + \ldots) \\
=& 10000(1.12682495 + \ldots) \\
\approx& 11268.25
\end{aligned}$$

In fact, adding the fifth term was not necessary to approximate A to the nearest dollar. It added only 5 cents to the estimate. The remaining terms will increase the estimate by less than a penny and need not be calculated. Thus, the final answer is $A \approx \$\boxed{11268}$.

7. The temperature measured in degrees Fahrenheit (F) is related to the temperature measured in degrees Celsius (C) by the formula $F = \frac{9}{5}C + 32$. At what temperature is the degrees Fahrenheit exactly twice the number of degrees Celsius? Express your answer in degrees Fahrenheit.

Solution: Replace F by $2C$ in the given equation and solve for C:

$$\begin{aligned} 2C &= \frac{9}{5}C + 32 \\ 2C - \frac{9}{5}C &= 32 \\ \frac{1}{5}C &= 32 \\ C &= 5 \cdot 32 = 160 \end{aligned}$$

Then $F = 2C = 2(160) = \boxed{320}^\circ$F.

This answer is easily checked using the conversion equation: $\frac{9}{5} \cdot 160 + 32 = 9 \cdot \frac{160}{5} + 32 = 9 \cdot 32 + 32 = 10 \cdot 32 = 320$.

8. Five regular polygons fit together like a puzzle; they share a vertex V and completely fill the plane around V without overlap. Find the largest possible sum of the measures of all of the angles of the five polygons, in degrees.

Solution: Note that the five angles with vertex V are the internal angles of the five regular polygons. Recall that the internal angle θ of a regular polygon with n sides is defined by $\theta = \frac{n-2}{n}180°$. If the number of sides of the regular polygon is $n = 3, 4, 5$ or 6 the internal angle θ is:

- $n = 3$ (triangle) $\theta = 60°$.
- $n = 4$ (square) $\theta = 90°$.
- $n = 5$ (pentagon) $\theta = 108°$.
- $n = 6$ (hexagon) $\theta = 120°$.

The measures of the five angles of the polygons must sum to 360° to fill the plane, and the average of the five angle measures is $360 \div 5 = 72°$. For the average to be 72, some angle measures will be less than 72° and some will be more than 72°. The only regular polygon with an angle measure less than 72° is the equilateral triangle (60°). Therefore proceed case by case, with each case defined by the number of regular polygons that are triangles.

The angle measures of five triangles sum to 300°, so all five polygons are not triangles. The angle measures of four triangles sum to 240°, which leaves 120° for the fifth polygon, which must be a hexagon. The angle measures of three triangles sum to 180°, which leaves 180° for two polygons. In this case, the other two polygons must be squares. The angle measures of two triangles sum to 120°, which leaves 240° for three polygons, which impossible since the angle measures must be at least 90°.

In summary, there are only two cases that work: four triangles and a hexagon ($60+60+60+60+120 = 360$), and three triangles and two squares ($60+60+60+90+90 = 360$). In the second case, the sum of all the angles in the polygons is $3 \cdot 180 + 2 \cdot 360 = 540 + 720 = 1260°$. In the first case, the sum of all the angles is $4 \cdot 180 + 720 = 720 + 720 = \boxed{1440}°$, which is the maximum possible sum of all the angles in the five polygons.

9. Find the value of the following infinite sum:

$$\sum_{n=2}^{\infty} \frac{1}{n^2+n}$$

Solution: The rational expression under the sum can be decomposed into two simpler fractions using the method of partial fractions:

$$\frac{1}{n^2+n} = \frac{1}{n} - \frac{1}{n+1}$$

Now replace the original argument of the summation with this difference of two rational expressions:

$$\sum_{n=2}^{\infty} \frac{1}{n^2+n} = \sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Now observe that the negative second fraction cancels the positive fraction for the next value of n at every iteration, so the sum of the first $N-1$ terms is:

$$\begin{aligned}
\sum_{n=2}^{N-1} \left(\frac{1}{n} - \frac{1}{n+1}\right) &= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{N-1} - \frac{1}{N}\right) \\
&= \frac{1}{2} + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) + \ldots + \left(-\frac{1}{N-1} + \frac{1}{N-1}\right) - \frac{1}{N} \\
&= \frac{1}{2} - \frac{1}{N}
\end{aligned}$$

A sum up to term $n = N-1$ therefore equals $\frac{1}{2} - \frac{1}{N}$. As $N-1$ increases without limit (to infinity), $\frac{1}{N}$ approaches 0 and the sum approaches $\boxed{\frac{1}{2}}$, the value of the infinite sum.